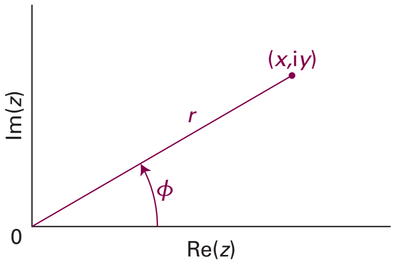
THE CHEMIST'S TOOLKIT 16 Euler's formula

A complex number $z = x + iy$ can be represented as a point in a plane, the **complex plane**, with Re(z) along the x-axis and Im(z) along the y-axis (Sketch 16.1). The position of the point can also be specified in terms of a distance r and an angle ϕ (the polar coordinates). Then $x = r\cos\phi$ and $y = r\sin\phi$, so it follows that

$$z = r(\cos\phi + i\sin\phi) \qquad (16.1)$$

The angle ϕ, called the **argument** of z, is the angle that r makes with the x-axis. Because $y/x = \tan\phi$, it follows that

$$r = (x^2 + y^2)^{1/2} = |z| \qquad \phi = \arctan\frac{y}{x} \qquad (16.2)$$



Sketch 16.1

One of the most useful relations involving complex numbers is **Euler's formula**:

$$e^{i\phi} = \cos\phi + i\sin\phi \qquad (16.3)$$

from which it follows that $z = r(\cos\phi + i\sin\phi)$ can be written

$$z = re^{i\phi} \qquad (16.4)$$

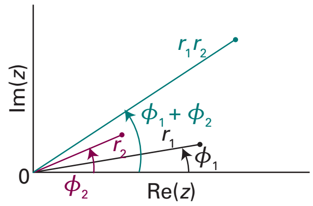
Two more useful relations arise by noting that $e^{-i\phi} = \cos(-\phi) + i\sin(-\phi) = \cos\phi - i\sin\phi$; it then follows that

$$\cos\phi = \tfrac{1}{2}(e^{i\phi} + e^{-i\phi}) \qquad \sin\phi = -\tfrac{1}{2}i(e^{i\phi} - e^{-i\phi}) \qquad (16.5)$$

The polar form of a complex number is commonly used to perform arithmetical operations. For instance, the product of two complex numbers in polar form is

$$z_1 z_2 = (r_1 e^{i\phi_1})(r_2 e^{i\phi_2}) = r_1 r_2 e^{i(\phi_1+\phi_2)} \qquad (16.6)$$

This construction is illustrated in Sketch 16.2.



Sketch 16.2

Brief illustration 16.1: Polar representation

Consider the complex number $z = 8 - 3i$. From *Brief illustration* 14.1, $r = |z| = 73^{1/2}$. The argument of z is

$$\phi = \arctan\left(\frac{-3}{8}\right) = -0.359 \text{ rad, or } -20.6°$$

The polar form of the number is therefore

$$z = 73^{1/2} e^{-0.359i}$$

Brief illustration 16.2: Roots

To determine the 5th root of $z = 8 - 3i$, note that from *Brief illustration* 16.1 its polar form is

$$z = 73^{1/2} e^{-0.359i} = 8.544 e^{-0.359i}$$

The 5th root is therefore

$$z^{1/5} = (8.544e^{-0.359i})^{1/5} = 8.544^{1/5} e^{-0.359i/5} = 1.536e^{-0.0718i}$$

It follows that $x = 1.536\cos(-0.0718) = 1.532$ and $y = 1.536\sin(-0.0718) = -0.110$ (note that the ϕ are in radians), so

$$(8 - 3i)^{1/5} = 1.532 - 0.110i$$