THE CHEMIST'S TOOLKIT 20 Angular momentum

Angular velocity, ω (omega), is the rate of change of angular position; it is reported in radians per second (rad s^{-1}). There are 2π radians in a circle, so 1 cycle per second is the same as 2π radians per second. For convenience, the 'rad' is often dropped, and the units of angular velocity are denoted s^{-1}.

Expressions for other angular properties follow by analogy with the corresponding equations for linear motion (*The chemist's toolkit* 3). Thus, the magnitude, J, of the **angular momentum**, J, is defined, by analogy with the magnitude of the linear momentum ($p = mv$):

$$J = I\omega \tag{20.1}$$

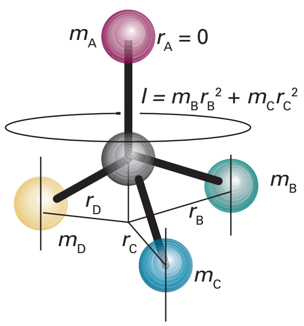
The quantity I is the **moment of inertia** of the object. It represents the resistance of the object to a change in the state of rotation in the same way that mass represents the resistance of the object to a change in the state of translation. In the case of a rotating molecule the moment of inertia is defined as

$$I = \sum_i m_i r_i^2 \tag{20.2}$$

where m_i is the mass of atom i and r_i is its perpendicular distance from the axis of rotation (Sketch 20.1). For a point particle of mass m moving in a circle of radius r, the moment of inertia about the axis of rotation is

$$I = mr^2 \tag{20.3}$$

The SI units of moment of inertia are therefore kilogram metre^2 (kg m^2), and those of angular momentum are kilogram metre^2 per second (kg m^2 s^{-1}).

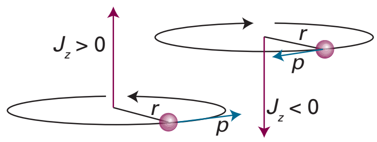


Sketch 20.1

The angular momentum is a vector, a quantity with both magnitude and direction (*The chemist's toolkit* 17). For rotation in three dimensions, the angular momentum has three components: J_x, J_y, and J_z. For a particle travelling on a circular path of radius r about the z-axis, and therefore confined to the xy-plane, the angular momentum vector points in the z-direction only (Sketch 20.2), and its only component is

$$J_z = \pm pr \tag{20.4}$$

where p is the magnitude of the linear momentum in the xy-plane at any instant. When $J_z > 0$, the particle travels in a clockwise direction as viewed from below; when $J_z < 0$, the motion is anticlockwise. A particle that is travelling at high speed in a circle has a higher angular momentum than a particle of the same mass travelling more slowly. An object with a high angular momentum (like a flywheel) requires a strong braking force (more precisely, a strong 'torque') to bring it to a standstill.



Sketch 20.2

The components of the angular momentum vector $\boldsymbol{J}$ when it lies in a general orientation are

$$J_x = yp_z - zp_y \qquad J_y = zp_x - xp_z \qquad J_z = xp_y - yp_x \tag{20.5}$$

where p_x is the component of the linear momentum in the x-direction at any instant, and likewise p_y and p_z in the other directions. The square of the magnitude of the vector is given by

$$J^2 = J_x^2 + J_y^2 + J_z^2 \tag{20.6}$$

By analogy with the expression for linear motion ($E_k = \frac{1}{2}mv^2 = p^2/2m$), the kinetic energy of a rotating object is

$$E_k = \tfrac{1}{2}I\omega^2 = \frac{J^2}{2I} \tag{20.7}$$

For a given moment of inertia, high angular momentum corresponds to high kinetic energy. As may be verified, the units of rotational energy are joules (J).

The analogous roles of m and I, of v and ω, and of p and J in the translational and rotational cases respectively provide a ready way of constructing and recalling equations. These analogies are summarized below:

Translation		Rotation	
Property	**Significance**	**Property**	**Significance**
Mass, m	Resistance to the effect of a force	Moment of inertia, I	Resistance to the effect of a twisting force (torque)
Speed, v	Rate of change of position	Angular velocity, ω	Rate of change of angle
Magnitude of linear momentum, p	$p = mv$	Magnitude of angular momentum, J	$J = I\omega$
Translational kinetic energy, E_k	$E_k = \frac{1}{2}mv^2 = p^2/2m$	Rotational kinetic energy, E_k	$E_k = \frac{1}{2}I\omega^2 = J^2/2I$