Differentiation IV

The product rule and the quotient rule

16

Answers to additional problems

16.1 The Planck function, $\frac{G}{T}$ [1]

This is a fraction and therefore a quotient.

16.2 The rate of reaction follows an equation of the type

$$\text{rate} = k\,c$$

during a first-order reaction. Here c is a concentration, k is the rate constant, and t is the time. [1]

The two terms are multiplied together, so a product.

16.3 The conductivity λ of an ion through a solution is a function of the mobility μ and the ion charge z,

$$\lambda = z\,F\,\mu$$ [1]

Three terms are multiplied together, so a product.

16.4 Rewriting the expression slightly, $\phi_{\text{atm}} = \frac{Z_i \exp(-kr)}{r} - \frac{Z_i}{r}$

where $k = 1/r_D$. The derivative of the second term, $-Z_i/r$ is simply Z_i/r^2.

Concerning the main function,

If $u = Z_i \exp(-kr)$ then $du/dx = -kZ_i \exp(-kr)$

If $v = r$ then $dv/dx = 1$

Inserting terms into the quotient rule yields,

$$\frac{d\phi_{\text{atm}}}{dr} = \frac{r[-kZ_i \exp(-kr)] - Z_i \exp(-kr)[1]}{r^2} + \frac{Z_i}{r^2}$$

Factorizing yields, $$\frac{d\phi_{\text{atm}}}{dr} = \frac{-Z_i\{kr+1\}\exp(-kr)}{r^2} + \frac{Z_i}{r^2}$$

Inserting for k yields, $$\frac{d\phi_{\text{atm}}}{dr} = \frac{-Z_i\left\{\left(r/r_D\right)+1\right\}\exp\left(r/r_D\right)}{r^2} + \frac{Z_i}{r^2}$$

16.5 **1.** If $u = \sin 2x$ then $du/dx = 2\cos 2x$

If $v = x^3$ then $du/dx = 3x^2$

Inserting terms, $$\frac{dy}{dx} = \frac{x^3[2\cos 2x] - \sin 2x[3x^2]}{(x^3)^2}$$

Factorizing yields, $$\frac{dy}{dx} = \frac{x^2\{2x\cos 2x - 3\sin 2x\}}{x^6} = \frac{2x\cos 2x - 3\sin 2x}{x^4}$$

2. If $u = \sin 2x$ then $du/dx = 2\cos 2x$

If $v = x^{-3}$ then $dv/dx = -3x^{-4}$

Inserting terms, $$\frac{dy}{dx} = \sin 2x[-3x^{-4}] + x^{-3}[2\cos 2x] = -\frac{3\sin 2x}{x^4} + \frac{2\cos 2x}{x^3}$$

In the second term, top and bottom are multiplied by x,

$$\frac{dy}{dx}=-\frac{3\sin 2x}{x^4}+\frac{2x\cos 2x}{x^4}$$

and placing over a common denominator of x^4 $\frac{dy}{dx}=-\frac{2x\cos 2x-3\sin 2x}{x^4}$

3. The two expressions in parts (1) and (2) are the same.

16.6 **1.** We obtain the derivative of exp $((a+b)x)$ straightforwardly using eqn. (14.1).

We say $\frac{dy}{dx}=(a+b)\exp((a+b)x)$

2. We require the product rule to differentiate $y=\exp(ax)\exp(bx)$,

If $u=\exp(ax)$ then $\frac{du}{dx}=a\exp(ax)$

If $v=\exp(bx)$ then $\frac{dv}{dx}=b\exp(bx)$

Inserting terms into eqn. (16.1) yields,

$$\frac{dy}{dx}=\exp(ax)\,[b\exp(bx)]+\exp(bx)\,[a\exp(ax)]$$

Factorizing yields, $(a+b)\,\{\exp(ax)\exp(bx)\}$

Equation (9.4) tells us that, $\exp(ax)\exp(bx)=\exp((a+b)x)$. Substituting for $\exp((a+b)x)$ in (1) yields (2).

So the results are the same.

16.7 The problem is a quotient.

If $u=4\xi^2$ then $du/d\xi=8\xi$

If $v=1-\xi^2$ then $dv/d\xi=-2\xi$

Inserting terms, $\frac{dK}{d\xi}=\frac{(1-\xi^2)[8\xi]-4\xi^2[-2\xi]}{(1-\xi^2)^2}$

Multiplying out the brackets, $\frac{dK}{d\xi}=\frac{8\xi-8\xi^3+8\xi^3}{(1-\xi^2)^2}$

So $\frac{dK}{d\xi}=\frac{8\xi}{(1-\xi^2)^2}$

16.8 The problem is a product.

We will need the **chain rule** to obtain the derivative of v.

If $u=AN^{-1/2}$ then $\frac{du}{dN}=\frac{1}{2}AN^{-3/2}=-\frac{AN^{-1/2}}{2}$ where $A=\left(\frac{2}{\pi}\right)^{1/2}$

if $v=\exp(BN^{-1})$ then $\frac{dv}{dN}=-BN^{-2}\exp(BN^{-1})$ where $B=-\frac{n^2}{2}$

Inserting terms, $\frac{dP}{dN}=AN^{-1/2}\left[-BN^{-2}\exp(BN^{-1})\right]+\exp(BN^{-1})\left[-\frac{AN^{-3/2}}{2}\right]$

The N term on the far right-hand side comes from $N^{-3/2}=N^{-1}\times N^{-1/2}$.

Factorizing yields, $\frac{dP}{dN}=AN^{-1/2}\exp(BN^{-1})\left\{\left[-BN^{-2}\right]-\left[\frac{N^{-1}}{2}\right]\right\}$

Re-inserting A and B terms,

$$\frac{dP}{dN}=\left(\frac{2}{\pi}\right)^{1/2}N^{-1/2}\left\{\left[-\left(-\frac{n^2}{2}\right)N^{-2}\right]-\left[\frac{N^{-1}}{2}\right]\right\}\exp\left(-\frac{n^2}{2}N^{-1}\right)$$

Finally, a little tidying, $\frac{dP}{dN}=\left(\frac{2}{\pi N}\right)^{1/2}\left\{\left[\frac{n^2}{2N^2}\right]-\left[\frac{1}{2N}\right]\right\}\exp\left(-\frac{n^2}{2N}\right)$

16.9 Here, $y=\frac{\sin\theta}{\cos\theta}$ which is a quotient,

If $u=\sin\theta$ then $\frac{du}{d\theta}=\cos\theta$

If $v = \cos\theta$ then $\dfrac{dv}{d\theta} = -\sin\theta$

Inserting terms into the quotient rule yields,

$$\frac{d(\tan\theta)}{d\theta} = \frac{\cos\theta[\cos\theta] - \sin\theta\,[\sin\theta]}{(\cos\theta)^2} = \frac{dy}{d\theta} = \frac{(\cos\theta)^2 + (\sin\theta)^2}{(\cos\theta)^2}$$

We can simplify this expression further because the top line is eqn. (11.13). $\sin^2\theta + \cos^2\theta = 1$ so,

$$\frac{d(\tan\theta)}{d\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta.$$

16.10 This problem is a product. For the purpose of this calculation, the central term $\exp\left(\frac{\Delta S^\ddagger}{R}\right)$ can be regarded as a constant, which we will call c.

If $u = \dfrac{k_B T c}{h}$ then $\dfrac{du}{dT} = \dfrac{k_B c}{h}$

If $v = \exp\left(-\dfrac{\Delta H^\ddagger}{R}T^{-1}\right)$ then $\dfrac{dv}{dT} = \dfrac{\Delta H^\ddagger}{R}T^{-2}\exp\left(-\dfrac{\Delta H^\ddagger}{R}T^{-1}\right)$

We need the **chain rule** to obtain the derivative *of v*.

Inserting terms into the product rule,

$$\frac{dk}{dT} = \left(\frac{k_B T c}{h}\right)\left[\frac{\Delta H^\ddagger}{R}T^{-2}\exp\left(-\frac{\Delta H^\ddagger}{R}T^{-1}\right)\right] + \left(\exp\left(-\frac{\Delta H^\ddagger}{R}T^{-1}\right)\right)\left[\frac{k_B c}{h}\right]$$

Factorizing,

$$\frac{dk}{dT} = \left(\frac{k_B c}{h}\right)\exp\left(-\frac{\Delta H^\ddagger}{R}T^{-1}\right)\left\{\left[T\left(\frac{\Delta H^\ddagger}{R}T^{-2}\right)\right] + 1\right\}$$

Further cancelling has simplified the *T* term in the final bracket.

Tidying yields,

$$\frac{dk}{dT} = \left\{1 + \frac{\cancel{T}\Delta H^\ddagger}{RT^{\cancel{2}}}\right\}\left(\frac{k_B c}{h}\right)\exp\left(-\frac{\Delta H^\ddagger}{RT}\right)$$

Finally, re-substituting for c,

$$\frac{dk}{dT} = \left\{1 + \frac{\Delta H^\ddagger}{RT}\right\}\left(\frac{k_B}{h}\right)\exp\left(-\frac{\Delta S^\ddagger}{R}\right)\exp\left(-\frac{\Delta H^\ddagger}{RT}\right)$$