Integration IV

Integrating areas and volumes, and multiple integration

22

Answers to additional problems

22.1 We obtain the entropy using the following integral, $\Delta S = \int_{T_1}^{T_2} \frac{C_p}{T}\,\mathrm{d}T$
Substituting for ΔS yields,

$$\Delta S = \int_{240}^{330}\left(\frac{91.47 + 7.5\times10^{-2}\ T}{T}\right)\mathrm{d}T = \int_{240}^{330}\left(\frac{91.47}{T} + 7.5\times10^{-2}\right)\mathrm{d}T$$

Integration yields, $\Delta S = [91.47 \ln T + 7.5\times10^{-2}\,T]_{240}^{330}$

Inserting the limits yields, $\Delta S = \left(91.47\ln\left(\frac{330}{240}\right) + 7.5\times10^{-2}\ (330-240)\right)$

so $\Delta S = (91.47 \times 0.318) + (7.5 \times 10^{-2} \times 90) = 29.13 + 6.75$

and $\Delta S = 35.88\ \mathrm{J\ K^{-1}\ mol^{-1}}$

22.2 **Strategy**

1. We find out where the two curves intersect.
2. We integrate each curve, using the two points of intersection as the limits.
3. We subtract the lesser area from the greater.

Solution

1. The first curve factorizes to form $y = (x^2 + 5x + 6)\ (x-4) = (x+2)\ (x+3)\ (x-4)$. The second curve factorizes to form $y = (x+2)\ (x-4)$ so the two curves intersect at the points $x = -2$ and at $x = 4$.
2. Upper curve $\int_{-2}^{4}\left(x^2 - 2x - 8\right)\mathrm{d}x = \left[\frac{x^3}{3} - x^2 - 8x\right]_{-2}^{4} = -26⅔ - (9⅓) = -36$ area units.

   Lower curve $\int_{-2}^{4}\left(x^3 + x^2 - 14x - 24\right)\mathrm{d}x = \left[\frac{x^4}{4} + \frac{x^3}{3} - 7x^2 - 24x\right]_{-2}^{4}$
   $= -122⅔ - (21⅓) = -144$ area units.
3. The area of overlap is, $144 - 36 = 108$ area units.
   - The areas beneath both curves are negative because each lies below the x-axis. But negative areas bear no relation to physical fact so from now on we will regard both as positive.

22.3 We obtain the area as the integral,

$$\int_1^3 y\,\mathrm{d}x = \left[\frac{4x^3}{3} - x^2 + 5x\right]_1^3 = \left(\frac{4}{3}\times27 - 9 + 15\right) - \left(\frac{4}{3} - 1 + 5\right) = 42 - 5\tfrac{1}{3} = 36\tfrac{2}{3}\ \text{area units.}$$

22.4 $V = \pi\int_0^3 x^2\,\mathrm{d}y$

Inserting the function yields, $V = \pi\int_0^3 \left(\frac{y}{4}\right)^{2/3}\mathrm{d}y$

Integration yields, $V = \frac{\pi}{4^{2/3}}\left[\frac{3}{5}y^{5/3}\right]_0^3$

Inserting limits yields, $V = \frac{\pi}{4^{2/3}} \times \frac{3}{5}\left(3^{5/3} - 0\right)$

so $V = \frac{\pi \times 3^{8/3}}{4^{2/3} \times 5} = 4.67$ volume units

22.5 We equate the two equations to find the coordinates where the two lines overlap. We say, $x^2 = -2x + 8$, so $0 = x^2 + 2x - 8$, so $0 = (x - 2)\ (x + 4)$. The two lines intersect at values of $x = -4$ and $+2$.

The area beneath the parabolic curve $= \int_{-4}^{2} x^2\ \mathrm{d}x = \left[\frac{x^3}{3}\right]_{-4}^{2} = 24$ area units.

The area beneath the line, $\int_{-4}^{2} -2x + 8\ \mathrm{d}x = [-x^2 + 8x]_{-4}^{2} = 60$ area units.

The area of overlap $= 60 - 24 = 36$ area units.

22.6 $\text{Volume} = \pi \int_2^4 y^2\, \mathrm{d}x$

Inserting the function yields, $\text{Volume} = \pi \int_2^4 (\exp(2x))^2\, \mathrm{d}x = \pi \int_2^4 \exp(4x)\mathrm{d}x$

Integration yields, $\text{Volume} = \pi \left[\frac{\exp(4x)}{4}\right]_2^4$

Inserting limits yields, $\text{Volume} = \frac{\pi}{4}(\exp(16) - \exp(8))$

so (to 4 s.f.) $\text{Volume} = \frac{\pi}{4}(8.886 \times 10^6 - 2.981 \times 10^3) = 6.977 \times 10^6$

22.7 **Strategy**

There are three separate integrals here, each embedded in the others.

1. As before, we integrate in three stages then multiply together the results of the three component answers.
2. We rearrange to equate the integral to 1.

Solution

1. From eqn. (22.3), $\int \psi^2 \mathrm{d}V = N^2 \int_0^\infty r^2 \exp\left(-\frac{2r}{a_0}\right)\mathrm{d}r \int_0^\pi \sin\theta\, \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi$
2. Integration with respect to ϕ. The relevant part of the outermost integral is,

   $$\int_0^{2\pi} \mathrm{d}\phi = 2\pi$$

   Integrate with respect to θ. Using a standard integral, the relevant part of the middle integral is, $\int_0^\pi (\sin\theta)\ \mathrm{d}\theta = \left[-\cos\theta\right]_0^\pi = -[(-1) - 1] = 2.$

   Integrate with respect to r. We need one of the standard integrals in Table 21.1. The relevant part of the innermost integral is,

   $$N^2 \int_0^\infty \left(r^2 \exp\left(-\frac{2r}{a_0}\right)\right)\mathrm{d}r = N^2 \times \tfrac{1}{4} a_0^3$$

3. The overall integral is $N^2 \times \tfrac{1}{4}a_0^3 \times 2 \times 2\pi = a_0^3 \pi N^2$.
4. We rearrange to make N the subject to ensure the integral in part 3 equals 1,

   $N = \left(\frac{1}{\pi a_0^3}\right)^{1/2}$, so the normalized wavefunction is,

   $$\psi = \left(\frac{1}{\pi a_0^3}\right)^{1/2} \exp\left(-\frac{r}{a_0}\right)$$

22.8 The respective integral is, $\int_0^1 \int_2^3 \int_0^1 8xyz\ \ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z$

It does not matter in which order we perform this integral because the functions are not interconnected. We will use the order x, then y, and finally z.

x-plane $\quad \int_0^1 8xyz\,\mathrm{d}x = \left[\frac{8x^2yz}{2}\right]_0^1 = \left[4x^2yz\right]_0^1 = 4yz$

y-plane $\quad \int_2^3 4yz\,\mathrm{d}y = \left[\frac{4y^2z}{2}\right]_2^3 = \left[2y^2z\right]_2^3 = 10z$

z-plane $\quad \int_0^1 10z\,\mathrm{d}z = \left[\frac{10z^2}{2}\right]_0^1 = \left[5z^2\right]_0^1 = 5$

The volume is therefore 5.

22.9 The probability $P = 1$ since the particle must lie somewhere on the surface of the sphere, therefore

$$\int_0^{2\pi}\int_0^{\pi} \psi^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = 1$$

Substituting for ψ, $\quad \int_0^{2\pi}\int_0^{\pi} N^2\cos^2\theta\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = 1$

We can separate out the different variables,

$$N^2\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\sin\theta\cos^2\theta\,\mathrm{d}\theta = N^2\left[\phi\right]_0^{2\pi}\left[\frac{\cos^3\theta}{-3}\right]_0^{\pi} = 1$$

Evaluating the integral, $\quad -\frac{N^2}{3}(2\pi)(\cos^3\pi - \cos^3 0) = 1$

Simplifying we find, $\quad -\frac{N^2}{3}(2\pi)((-1)^3 - 1^3) = 1$

This becomes, $\quad N^2\left(\frac{4\pi}{3}\right) = 1$

so we can rearrange this expression to find the normalization constant, $N = \sqrt{\frac{3}{4\pi}}$.

22.10 Substituting in for the two wavefunctions,

$$\int_0^{2\pi}\int_0^{\pi} N_1\cos\theta \times N_2\sin\theta\,\mathrm{e}^{\mathrm{i}\phi} \times \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

Separating the variables gives,

$$N_1N_2\int_0^{2\pi}\mathrm{e}^{\mathrm{i}\phi}\mathrm{d}\phi\int_0^{\pi}\cos\theta\times\sin^2\theta\,\mathrm{d}\theta$$

Evaluating the integrals,

$$N_1N_2\left[\frac{\mathrm{e}^{\mathrm{i}\phi}}{\mathrm{i}}\right]_0^{2\pi}\left[\frac{\sin^3\theta}{3}\right]_0^{\pi} = \frac{N_1N_2}{3\mathrm{i}}(\mathrm{e}^{2\pi\mathrm{i}} - 1)(\sin^3\pi - \sin^3 0)$$

Both sin terms equal 0, so we conclude that,

$$\int_0^{2\pi}\int_0^{\pi}\psi_1\psi_2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = 0$$

This value of 0 means they are indeed orthogonal.