Complex numbers

25

Answers to additional problems

25.1

$$\mathbf{I}_x\mathbf{I}_y = \frac{1}{2}\times\frac{1}{2\mathrm{i}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = \frac{1}{4\mathrm{i}}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$$

$$\mathbf{I}_y\mathbf{I}_x = \frac{1}{2\mathrm{i}}\times\frac{1}{2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \frac{1}{4\mathrm{i}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

so $$\mathbf{I}_x\mathbf{I}_y - \mathbf{I}_y\mathbf{I}_x = \frac{1}{4\mathrm{i}}\begin{pmatrix}-2 & 0\\ 0 & 2\end{pmatrix} = -\frac{2}{4\mathrm{i}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

The factor of 2/4 cancels to ½. We then manipulate to remove the i from the denominator, saying i/i = 1 ,

$$\mathbf{I}_x\mathbf{I}_y - \mathbf{I}_y\mathbf{I}_x = -\frac{\mathrm{i}}{2\mathrm{i}^2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = +\frac{\mathrm{i}}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \mathrm{i}\times\mathbf{I}_z$$

25.2 $$\mathbf{I}_y\mathbf{I}_z = \frac{1}{4i}\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix} \qquad I_zI_y = \frac{1}{4\mathrm{i}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

so $$\mathbf{I}_y\mathbf{I}_z - \mathbf{I}_z\mathbf{I}_y = \frac{1}{4i}\left\{\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix} - \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right\} = \frac{1}{4\mathrm{i}}\begin{pmatrix}0 & -2\\ -2 & 0\end{pmatrix} = -\frac{2}{4\mathrm{i}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

Again, we cancel the 2 and 4 to yield ½. Manipulation to remove i from the denominator gives,

$$\mathbf{I}_y\mathbf{I}_z - \mathbf{I}_z\mathbf{I}_y = -\frac{\mathrm{i}}{2\mathrm{i}^2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \frac{\mathrm{i}}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \mathrm{i}\times I_x$$

25.3 $$\mathbf{I}_z\mathbf{I}_x - \mathbf{I}_x\mathbf{I}_z = \frac{1}{4}\left\{\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} - \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right\} = \frac{1}{4}\begin{pmatrix}0 & 2\\ -2 & 0\end{pmatrix}$$

Factorizing yields $= \frac{2}{4}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$

Again, we can cancel the 2 and the 4 to yield ½.

The matrix should remind us of the matrix in the expression for $\mathbf{I}_y$, but with a slightly different factor. But if we multiply by i/i = 1, we generate i times $\mathbf{I}_y$,

$$\frac{\mathrm{i}}{2\mathrm{i}}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = \mathrm{i}\times\frac{1}{2\mathrm{i}}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = \mathrm{i}\times\mathbf{I}_y$$

25.4 $$\frac{1}{Z_{\text{total}}} = \frac{1}{Z_c} + \frac{1}{Z_{R_1}} + \frac{1}{Z_{R_2}} = \frac{1}{1/\mathrm{i}\omega C} + \frac{1}{R_1} + \frac{1}{R_2}$$

$$\frac{1}{Z_{\text{total}}} = \frac{R_1R_2\mathrm{i}\omega C + (R_1 + R_2)}{R_1R_2}$$

so $$Z_{\text{total}} = \frac{R_1R_2}{R_1R_2\mathrm{i}\omega C + (R_1 + R_2)}$$

25.5 12 – 6i.

25.6 A general square ($x^2 + y^2$) has roots ($x + y$i) ($x - y$i). Therefore, ($3a + 7b$i) ($3a - 7b$i).

25.7 Using the formula in eqn. (25.11),

$e^{ikx} = \cos kx + \mathrm{i}\sin kx$ and

$e^{-ikx} = \cos(-kx) + \mathrm{i}\sin(-kx) = \cos(kx) - \mathrm{i}\sin(kx)$

Therefore, $\psi = Ae^{ikx} + Be^{-ikx}$ can be rewritten as,

$\psi = A(\cos kx + \mathrm{i}\sin kx) + B(\cos(kx) - \mathrm{i}\sin(kx))$

$\psi = (A + B)\cos kx + (A - B)\mathrm{i}\sin kx$

25.8 Multiply together the two wavefunctions Ψ and Ψ*, so multiply the brackets in the following problem,

$\Psi\Psi^* = (f + \mathrm{i}g)(f - \mathrm{i}g)$

We simplify this problem by recognizing how the two brackets resemble the factors of the difference of two squares.

Therefore, the product $\Psi\Psi^* = (f^2 - \mathrm{i}^2g^2) = (f^2 + g^2)$.

It is incorrect to factor out the i, as $\Psi\Psi^* = \mathrm{i}(f + g)(f - g)$ because the two f terms were not originally multiplied by i.

25.9 Classically, we know that the kinetic energy E_{KE} is,

$E_{\text{KE}} = ½\, mv^2$

$$E_{\text{KE}} = \frac{mv^2}{2} \times \frac{m}{m} = \frac{(mv)^2}{2m} = \frac{p^2}{2m}$$

Therefore, substituting in for the w momentum operator, we can derive the kinetic energy operator,

$$E_{\text{KE}} = \frac{p^2}{2m}$$

$$E_{\text{KE}} = \frac{-\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}x} \times -\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}x}}{2m}$$ (notice the way that, –i × –i = –1)

$$E_{\text{KE}} = \frac{(-1)^2(\mathrm{i})^2\hbar^2\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)}{2m}$$

so $$E_{\text{KE}} = \frac{-\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$$

This operator appears in the one-dimensional Schrödinger equation,

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + V\Psi = E\Psi$$

25.10 We know from eqn. (25.13) that,

$e^{ikx} = (\cos kx + i \sin kx)$

and $e^{-ikx} = (\cos(-kx) + i \sin(-kx)) = (\cos kx - i \sin kx)$.

Therefore, we can rewrite the wavefunction as,

$\psi = A(e^{ikx} \pm e^{-ikx})$

$\psi = A((\cos kx + i \sin kx) \pm (\cos kx - i \sin kx))$

$\psi = A((\cos kx \pm \cos kx) + i(\sin kx \mp \sin kx))$

There are two possible solutions,

- If we take the top line of the ± symbols, then $\psi = 2A \cos kx$.
- If we take the lower line of the ± symbols, then $\psi = 2iA \sin kx$.
- The next step is to see which of these forms of the wavefunction best satisfies the boundary conditions.